Introduction to the theta correspondence

Kevin Chang

June 15, 2021

Contents

1 Introduction 1
  1.1 Notation and conventions 2

2 The Weil representation and the theta correspondence 3
  2.1 The Weil representation over a $p$-adic field 3
  2.2 The Weil representation over a finite field 5
  2.3 Main example: The theta correspondence for $(\text{Sp}(2), \text{O}^+(4))$ 6

3 The preservation principle 7
  3.1 The preservation principle for the local theta correspondence 7
  3.2 Main example: A chain of supercuspidals 8
  3.3 The preservation principle over a finite field 9
  3.4 Main example: The preservation principle for $(\text{Sp}(2), \text{O}^+(4))$ 9

4 Compatibility 10
  4.1 A result of Pan 10
  4.2 Main example: Compatibility between $p$-adic $(\text{Sp}(V_2), \text{O}(V_6^+))$ and finite $(\text{SL}(2), \text{O}^-(2) \times \text{O}^+(4))$ 11
  4.2.1 Components are not cuspidal 11
  4.2.2 Components are cuspidal 13

5 Continuation of the main example 13

1 Introduction

As every Hollywood executive knows, if there’s ever a good thing, it needs to be milked dry. The local theta correspondence is a powerful method for producing representations of $p$-adic groups following precisely this philosophy. In slightly less vague terms, the local theta correspondence is a matching between certain representations of certain pairs of $p$-adic reductive groups known as dual reductive pairs. If we have a chain of $p$-adic groups with each consecutive pair a
dual reductive pair and a representation of the first group in the chain, we can try to produce a chain of representations by repeatedly applying the local theta correspondence.

The local theta correspondence is especially useful for the study of supercuspidal representations, which are especially difficult to construct. The preservation principle of Howe and Kudla is a recipe for picking our chain of $p$-adic reductive groups such that two miracles occur: (1) the chain of representations produced by the local theta correspondence can be extended forever (2) every representation in it is supercuspidal.

There is an analogue of the local theta correspondence for dual reductive pairs over finite fields. We call this the finite theta correspondence. Although the finite theta correspondence is not as nice as the local theta correspondence, it enjoys a variant of the preservation principle due to Pan. Another powerful tool for producing representations of finite reductive groups is Deligne-Lusztig induction, and as may be expected, the finite theta correspondence and Deligne-Lusztig induction are related in nice ways.

This paper is an exposition of several different topics related to the theta correspondence, covering both the local theta correspondence and the finite theta correspondence. For concreteness, we also work out several relatively well-understood examples, which recur throughout this paper. In Section 2, we review the construction of the Weil representation and introduce the theta correspondence. We also state a formula of Srinivasan and Pan that explicitly decomposes the uniform part of the Weil representation for certain finite dual reductive pairs in terms of Deligne-Lusztig virtual characters. In Section 3, we state the preservation principle and explain its use for constructing chains of supercuspials and cuspidals. In Section 4, we discuss a result of Pan that establishes some compatibility between the local theta correspondence and the finite theta correspondence. In Section 5, we discuss some proposed steps for extending our methods from the well-understood examples to an entire chain of groups.

1.1 Notation and conventions

Throughout, we use $F$ to denote a fixed $p$-adic field with $p$ odd, we use $\mathfrak{f}$ to denote its residue field, and we use $\varpi$ to denote a fixed uniformizer of $F$. We use 1 to denote the trivial representation and $\text{sgn}$ to denote the sign representation for orthogonal groups.

If $V$ (resp. $\mathbf{v}$) is an $F$-vector space (resp. $\mathfrak{f}$-vector space) with a bilinear form, we let $G(V)$ denote the subgroup of $\text{GL}(V)$ (resp. $\text{GL}(\mathbf{v})$) that preserves the bilinear form. We will mainly use $G(V)$ and $G(\mathbf{v})$ to denote orthogonal and symplectic groups.

Any special orthogonal group $SO(V)$ (over any field of odd characteristic) has a unique automorphism given by conjugation by any element of its complement in $O(V)$. We will denote this automorphism $\sigma$ throughout this paper.
The Weil representation and the theta correspondence

We briefly review the Weil representation and the theta correspondence, both in the $p$-adic field and finite field setting.

2.1 The Weil representation over a $p$-adic field

The primary reference for this section is the first two chapters of [4].

We first review the construction of the Weil representation. Let $\psi$ be an additive character of $F$, and let $(W, \langle , \rangle)$ be a symplectic $F$-vector space. The Heisenberg group $H(W) := W \oplus F$ has multiplication

$$(w_1, t_1)(w_2, t_2) = \left( w_1 + w_2, t_1 + t_2 + \frac{1}{2}\langle w_1, w_2 \rangle \right).$$

By the Stone-von Neumann, there exists a unique smooth irreducible representation $(\rho_\psi, S)$ of $H(W)$ with central character $\psi$, i.e. $F \subset H(W)$ acts by $\psi$.

Note that $\text{Sp}(W)$ has a right action on $H(W)$:

$$(w, t)g = (g^{-1}w, t).$$

We can twist $(\rho_\psi, S)$ by this action: for any element $g \in \text{Sp}(W)$, let $(\rho_g^\psi, S)$ be defined by $\rho_g^\psi(h) = \rho_\psi(hg)$. Since the action of $\text{Sp}(W)$ on the center $F \subset H(W)$ is trivial, $\rho_g^\psi$ is a smooth irreducible representation of $H(W)$ with central character $\psi$. By the uniqueness of $(\rho_\psi, S)$, there is an automorphism $A(g) : S \to S$ such that $A(g)^{-1}\rho_\psi(h)A(g) = \rho_g^\psi(h)$. This automorphism is unique up to scaling by an element of $C^\times$, since by Schur’s lemma, the only automorphisms of $(\rho_\psi, S)$ are given by scalar multiplication. Thus, we have a projective representation of $\text{Sp}(W)$ on $S$.

The projective representation of $\text{Sp}(W)$ becomes a bona fide representation when we pull it back to a double cover $\tilde{\text{Sp}}(W) \to \text{Sp}(W)$ called the metaplectic group. We refer to this representation as the Weil representation (determined by the additive character $\psi$).

Having a representation of $\text{Sp}(W)$ is useful for pairing representations of dual reductive pairs, which are pairs of reductive subgroups of $\text{Sp}(W)$ that are mutual centralizers. More precisely, let $V$ and $V'$ be $F$-vector spaces equipped with bilinear forms. Then a pair of subgroups $G, G' \subset \text{Sp}(W)$ is a dual reductive pair if $G$ and $G'$ are reductive, $G = C_{\text{Sp}(W)}(G')$, and $G' = C_{\text{Sp}(W)}(G)$. If $(G, G')$ is a dual reductive pair, then their preimages $\tilde{G}, \tilde{G}' \subset \tilde{\text{Sp}}(W)$ centralize each other, so we can pull back $\omega_\psi$ along the map $\tilde{G} \times \tilde{G}' \to \tilde{\text{Sp}}(W)$.

For simplicity (and due to the author’s lack of knowledge), for many of the later results, we will restrict to dual reductive pairs of the form $(\text{Sp}(V), \text{O}(V'))$, ...
where $V$ is a symplectic $F$-vector space, $V'$ is an even-dimensional quadratic $F$-vector space, and $V \otimes V' = W$. Other good examples for which similar results (to the ones we will describe later) hold include pairs of unitary groups and symplectic/odd-orthogonal pairs (these are called pairs of type I). One reason that symplectic/even orthogonal pairs are convenient is that the maps $\widetilde{\text{Sp}}(V) \to \text{Sp}(V)$ and $\widetilde{\text{O}}(V') \to \text{O}(V')$ admit sections. Thus, once we pick sections, the Weil representation $\omega_\psi$ pulls back to a representation of $\text{Sp}(V) \times \text{O}(V')$. Sometimes, the choice of sections matters; we will say if this is the case. When it does not, we will implicitly fix choices of sections for every symplectic/even orthogonal pair throughout.

Of course, there are dual reductive pairs that are not symplectic/even orthogonal, but there does exist a classification of $p$-adic dual reductive pairs (see [8, Chapter 1]). The irreducible dual reductive pairs are as follows (up to switching the order of the groups):

1. $(U_\epsilon(V), U^{-\epsilon}(V'))$: Let $D$ be either $F$, a quadratic extension of $F$, or a division quaternion algebra over $F$. Fix $\epsilon = \pm 1$. Here, $V$ is an $\epsilon$-skew Hermitian right $D$-vector space, and $V'$ is a $-\epsilon$-skew Hermitian left $D$-vector space.

2. $(\text{GL}_D(V), \text{GL}_D(V'))$: Here, $D$ is a central division algebra over $F$, and $V$ and $V'$ are right vector spaces over $D$.

Pairs of types (a), such as symplectic/orthogonal pairs (take $D = F$ and $\epsilon = -1$), are said to be of type I, and pairs of type (b) are said to be of type II. Type I pairs are embedded into symplectic groups in roughly the same way as symplectic/orthogonal pairs are. Since we will not discuss type II pairs further, we will not say how they are embedded into symplectic groups.

For any dual reductive pair $(G, G')$ in $\text{Sp}(W)$, we can match representations of $G$ and representations of $G'$ using the Weil representation. More precisely, we say that an irreducible admissible $G$-representation $\pi$ corresponds to an irreducible admissible $G'$-representation $\pi'$ if there is a nontrivial $G \times G'$-map $\omega_\psi \to \pi \otimes \pi'$. This correspondence $\pi \mapsto \theta(\pi, V')$ is called the local theta correspondence, and it was proven by Howe and Waldspurger to be 1-1 (see [13]). Note that in the case we care about (symplectic/even orthogonal), the local theta correspondence matches representations of $\text{Sp}(V)$ and representations of $\text{O}(V')$, due to the fact that the Weil representation pulls back to a representation of $\text{Sp}(V) \times \text{O}(V')$.

As a sidenote, the fact that the local theta correspondence is 1-1 in the case of residue characteristic 2 was proven much more recently in [7], [3], and [2].

**Remark 1.** For convenience, we allow groups in a dual reductive pair to be trivial, even if this does not quite agree with the definition. The theta correspondence when one of the groups is trivial just pairs the trivial representation with the trivial representation.
2.2 The Weil representation over a finite field

We can define the Weil representation, dual reductive pairs, and the theta correspondence over finite fields in the exact same way as the $p$-adic field case. Unlike the local theta correspondence, the theta correspondence for finite dual reductive pairs is known not to be 1-1. On the bright side, in the finite field setting, the Weil representation descends to a representation of the symplectic group, so we do not need to worry about metaplectic covers. Moreover, the Weil representation for $\text{Sp}(2n)$ has dimension $q^n$, and finite-dimensional representations are easier to deal with than infinite-dimensional ones.

Although the entire Weil representation is quite mysterious, if we restrict our attention to the part coming from Deligne-Lusztig induction (known as the uniform part), the Weil representation over a finite field can be decomposed explicitly for certain dual reductive pairs, including the symplectic/even orthogonal pairs we are interested in. Given a connected finite reductive group $G$, let $C := C(G)$ denote the space of complex-valued class functions on $G$, and let $U$ denote the subspace spanned by the Deligne-Lusztig characters $R^G_T(\theta)$, where $T$ and $\theta$ range over the tori of $G$ and their characters, respectively. Given a class function $f \in C$, its uniform part is the projection of $f$ to $U$. If $G$ is a product of connected finite reductive groups $G' \times G''$, the uniform part of $f$ is the same as the tensor product of the projections onto the uniform parts with respect to $G'$ and $G''$.

In [12], for $|f|$ sufficiently large, Srinivasan decomposes the uniform part of the Weil representation for a symplectic/even orthogonal pair into Deligne-Lusztig characters. In [10], Pan extends it to all $f$ (of odd characteristic). To state the decomposition, we first need the following lemma from [12]. Here, $O^+(2k)$ (resp. $O^-(2k)$) denotes the isometry group of the quadratic form with matrix $\text{diag}(1, -1, 1, -1, \ldots, 1)$ (resp. $\text{diag}(1, -1, 1, -1, \ldots, 1, -d)$, where $d$ is a non-square). These are the only two isomorphism classes of quadratic forms on a $2k$-dimensional $\mathbb{F}$-vector space.

Lemma 1 (Srinivasan). There is a natural bijection between the set of conjugacy classes of maximal tori in $\text{Sp}(2k)$ and the union of the set of $O^+(2k)$-conjugacy classes of maximal tori in $\text{SO}^+(2k)$ and the set of $O^-(2k)$-conjugacy classes of maximal tori in $\text{SO}^-(2k)$.

The result is stated for symplectic/even special orthogonal pairs rather than symplectic/even orthogonal pairs because Deligne-Lusztig induction applies to connected groups. Here, if $L$ is a Levi subgroup of a parabolic subgroup $P \subset G$ ($L$ must be defined over $\mathbb{F}_q$, but $P$ does not need to be) and $\theta$ is a 1-dimensional representation of $L$, then we have a virtual representation $R^G_L(\theta) := \frac{1}{|W_L(T)|} \sum_{T \subset L} R^G_T(\theta)$, where the sum is taken over conjugacy classes of maximal tori in $L$.

Theorem 1 (Srinivasan, Pan). Let $\omega$ be the restriction of the Weil representation of $\text{Sp}(2n) \times \text{SO}^+(2m)$, and let $\omega_{\text{unif}}$ denote its uniform part. Then we have the following decompositions. Here, we identify conjugacy classes of maximal...
tori \( T \subset \text{Sp}(2k) \) with conjugacy classes of maximal tori \( T \subset \text{SO}^{±}(2k) \) using Lemma [1] and \( W(T) \) denotes the Weyl group of \( T \) in \( \text{Sp}(2k) \). For a torus \( T \subset \text{Sp}(2k) \), the sign \( \epsilon' \) is such that \( T \subset \text{SO}^{\epsilon'}(2k) \).

(a) \( n < m \):

\[
\omega_{\text{unif}} = \sum_{k=0}^{n} \sum_{(T) \subset \text{Sp}(2k)} \frac{1}{|W(T)|} \sum_{\theta \in T} \epsilon' R_{T \times \text{Sp}(2n-2k)}^{\text{Sp}(2n)}(\theta \otimes 1) \otimes R_{T \times \text{SO}^{\epsilon'}(2m)}^{\text{SO}^{\epsilon'}(2m-2k)}(\theta \otimes 1).
\]

(b) \( n \geq m \):

\[
\omega_{\text{unif}} = \sum_{k=0}^{m-1} \sum_{(T) \subset \text{Sp}(2k)} \frac{1}{|W(T)|} \sum_{\theta \in T} \epsilon' R_{T \times \text{Sp}(2n-2k)}^{\text{Sp}(2n)}(\theta \otimes 1) \otimes R_{T \times \text{SO}^{\epsilon'}(2m)}^{\text{SO}^{\epsilon'}(2m-2k)}(\theta \otimes 1)
+ \sum_{(T) \subset \text{Sp}(2m)} \epsilon(-1)^{n+m} \frac{2}{|W(T)|} \sum_{\theta \in T} R_{T \times \text{Sp}(2n-2m)}^{\text{Sp}(2n)}(\theta) \otimes R_{T}^{\text{SO}^{\epsilon'}(2m)}(\theta).
\]

We are interested in symplectic/even orthogonal pairs, so we briefly recall the relationship between representations of \( \text{O}^{\epsilon}(2m) \) and \( \text{SO}^{\epsilon'}(2m) \) (this all applies to representations of a group and a subgroup of index 2). Let \( \rho \) be an irreducible representation of \( \text{SO}^{\epsilon'}(2m) \). Recall that \( \sigma \) denotes conjugation by an element of \( \text{O}^{\epsilon}(2m) \) outside \( \text{SO}^{\epsilon'}(2m) \). If \( \rho = \sigma(\rho) \), then \( \text{Ind}_{\text{SO}^{\epsilon'}(2m)}^{\text{O}^{\epsilon}(2m)}(\rho) \) decomposes as a direct sum of irreducible \( \text{O}^{\epsilon}(2m) \)-representations \( \rho' \oplus \rho^{\sigma} \), where \( \rho' \otimes \text{sgn} \cong \rho^{\sigma} \).

If \( \rho \neq \sigma(\rho) \), then \( \text{Ind}_{\text{SO}^{\epsilon'}(2m)}^{\text{O}^{\epsilon}(2m)}(\rho) \) is an irreducible \( \text{O}^{\epsilon}(2m) \)-representation.

### 2.3 Main example: The theta correspondence for \((\text{Sp}(2), \text{O}^{+}(4))\)

We can use Theorem [1] to deduce part of the theta correspondence for the pair \((\text{Sp}(2), \text{O}^{+}(4))\). Since \( \text{Sp}(2) = \text{SL}(2) \), we will just use \( \text{SL}(2) \) from now on. We first review some of the representation theory of \( \text{SL}(2) \), which is well-understood. There is a 1-dimensional nonsplit torus \( \mu_{q+1} \subset \text{SL}(2) \) equal to the norm 1 subgroup of the quadratic extension \( \mathbb{F} \) (here, we identify \( \mathbb{F}^2 \) with \( \mathbb{C} \)). The characters \( \theta \) of \( \mu_{q+1} \) give rise to Deligne-Lusztig characters \( R'(\theta) := R_{\text{SL}(2)}^{\mu_{q+1}}(\theta) \), where \( R'(\theta) \cong R'(\theta^{-1}) \). When \( \theta^2 \neq 1 \), we say that \( \theta \) is regular.

More generally, a character of a maximal torus in a finite reductive group is regular if it has trivial stabilizer under the action of the Weyl group of the torus. In our situation, the Weyl group of \( \mu_{q+1} \) is \( \mathbb{Z}/2 \), acting on \( \mu_{q+1} \) by inversion, so \( \theta^2 \neq 1 \) if and only if \( \theta \) is regular. For regular characters \( \theta, R'(\theta) \) is an irreducible cuspidal representation of \( \text{SL}(2) \) (this is still true up to a sign for general connected reductive groups). Thus, we obtain \( 2^{-1} \) distinct irreducible cuspidal characters of \( \text{SL}(2) \) from the regular characters of \( \mu_{q+1} \). For details, see [1].
Let \( \theta \) be a regular character of \( \mu_{q+1} \). Since \( SO^{-}(2) \cong \mu_{q+1} \), the formula in Theorem 3.1 pairs \( R'(\theta) \cong R'(\theta^{-1}) \) with the virtual character of \( SO^{+}(4) \)

\[
-\frac{1}{2} \left( R^{SO^{+}(4)}_{SO^{-}(2) \times SO^{-}(2)}(\theta \otimes 1) + R^{SO^{+}(4)}_{SO^{-}(2) \times SO^{-}(2)}(\theta^{-1} \otimes 1) \right).
\]

Thus, there is some irreducible constituent \( \rho \) of \( R^{SO^{-}(2) \times SO^{-}(2)}_{SO^{-}(2)}(\theta \otimes 1) \) such that \( R'(\theta) \otimes \rho \) occurs in the Weil representation for \( (SL(2), SO^{+}(4)) \). If \( \rho = \sigma(\rho) \), then at least one of \( R'(\theta) \otimes \rho' \) or \( R'(\theta) \otimes \rho^{II} \) occurs in the theta correspondence for \( (SL(2), O^{+}(4)) \). If \( \rho \neq \sigma(\rho) \), then \( R'(\theta) \otimes \text{Ind}_{SO^{+}(4)}^{O^{+}(4)}(\rho) \) occurs in the theta correspondence for \( (SL(2), O^{+}(4)) \).

### 3 The preservation principle

Both the local theta correspondence and the finite theta correspondence enjoy a property known as the preservation principle, which describes “first occurrences” of supercuspidals in the local theta correspondence and cuspidals in the finite theta correspondence.

#### 3.1 The preservation principle for the local theta correspondence

For a fixed anisotropic quadratic or symplectic \( F \)-vector space \( V_{an}' \), we consider the **Witt tower**

\[
V_{an}' \subset V_{an} \oplus V_{1,1}' \subset V_{an} \oplus V_{2,2}' \subset \cdots,
\]

where \( V_{k,\ell}' \) is a direct sum of \( k \) hyperbolic planes.

If \( (G(V), G(V_{an}') \) is a dual reductive pair, then so are all the pairs \( (G(V), G(V_{an} \oplus V_{k,\ell}') \). Then the theta correspondences of these pairs are compatible in the following sense (cf. [3]).

**Proposition 1** (Howe, Kudla). Let \( \pi \) be an irreducible admissible representation of \( G(V) \).

(a) **(Persistence principle)** If \( \theta(\pi, V_{an}' \oplus V_{k_0,\ell_0}') \neq 0 \), then \( \theta(\pi, V_{an} \oplus V_{k,\ell}') \neq 0 \) for all \( k \geq k_0 \).

(b) **(Stable range)** For any \( k \geq \dim V \), \( \theta(\pi, V_{an} \oplus V_{k,\ell}') \neq 0 \).

(c) **(First occurrence of supercuspidal representations)** Suppose \( \pi \) is supercuspidal and \( k_0 \) is such that \( \theta(\pi, V_{an} \oplus V_{k_0-1,\ell_0-1}') = 0 \) and \( \theta(\pi, V_{an} \oplus V_{k_0,\ell_0}') \neq 0 \). Then \( \pi' := \theta(\pi, V_{an}' \oplus V_{k_0,\ell_0}') \) is supercuspidal, and \( \theta(\pi, V_{an}' \oplus V_{k,k}') \) is not supercuspidal for any \( k > k_0 \).

The correspondence \( \pi \leftrightarrow \pi' \) in Proposition 1(c) is called a **first occurrence of supercuspidal representations**, and it can be used to build chains of supercuspidal representations using the following result from [6], which comprises several cases of the **preservation principle**.
Theorem 2 (Kudla, Rallis). Let $\pi$ be an irreducible supercuspidal representation of $G(V)$, and let $\{V_{an}^{+} + V_{k,k}^{0} | k \geq 0\}$ and $\{V_{an}^{-} + V_{k,k}^{0} | k \geq 0\}$ be two Witt towers. Let $n^{\pm}(\pi)$ denote the smallest dimension of $V_{an}^{\pm} + V_{k,k}^{0}$ such that $\pi$ occurs in the theta correspondence for $(G(V), G(V_{an}^{\pm} + V_{k,k}^{0}))$.

(a) If $G(V)$ is symplectic and $V_{an}^{+}$ and $V_{an}^{-}$ are quadratic spaces of dimension 2 with the same determinant but different Hasse invariants, then $n^{+}(\pi) + n^{-}(\pi) = 2 \dim V + 4$.

(b) If $G(V)$ is even orthogonal and $V_{an}^{+}$ and $V_{an}^{-}$ are symplectic with dimension 0, then $n^{+}(\pi) + n^{-}(\pi \otimes \text{sgn}) = 2 \dim V$.

3.2 Main example: A chain of supercuspidals

Using the preservation principle (Theorem 2), we can build a chain of supercuspidals starting from any supercuspidal of a symplectic or even orthogonal group. For concreteness, we consider the following sequence of groups, alternating orthogonal and symplectic:

$$O(V_{2+2k}^{\pm}), Sp(W_2), O(V_{10}^{-}), Sp(W_{10}), O(V_{26}^{\pm}), Sp(W_{26}), O(V_{38}), \ldots$$

Here, the subscripts denote the dimensions of the spaces, and the signs on the quadratic spaces denote the two Witt towers running parallel to each other ($V_{2+2k} = V_{2}^{\pm} + V_{k,k}$).

For extra concreteness, we let $V_{2}^{+}$ and $V_{2}^{-}$ (the first spaces in the Witt towers) both be the unique unramified quadratic extension $E/F$. We set the quadratic form on $V_{2}^{+}$ to $\text{Nm}_{E/F}$ and the quadratic form on $V_{2}^{-}$ to $\varpi \text{Nm}_{E/F}$. Then the groups $O(V_{2}^{\pm})$ (these are the same group because preserving $\text{Nm}_{E/F}$ is equivalent to preserving $\varpi \text{Nm}_{E/F}$) are especially easy to describe. In our situation, $SO(V_{2}^{\pm})$ is isomorphic to the kernel of $\text{Nm}_{E/F}: E^\times \to F^\times$.

Now we construct our chain of supercuspidals. We start with an irreducible representation $\pi_0$ of $O(V_{2}^{+})$ other than the trivial representation and $\text{sgn}$. All nontrivial irreducible representations of $O(V_{2}^{+})$ are supercuspidal because $SO(V_{2}^{+})$ has no proper parabolic subgroups. Thus, the preservation principle implies that $n^{+}(\pi_0) + n^{-}(\pi_0 \otimes \text{sgn}) = 4$. Since $\pi_0$ and $\pi_0 \otimes \text{sgn}$ are nontrivial, we cannot have $n^{+}(\pi_0) = 0$ or $n^{-}(\pi_0 \otimes \text{sgn}) = 0$. Thus, $n^{+}(\pi_0) = n^{-}(\pi_0 \otimes \text{sgn}) = 2$.

Let $\pi_1$ be the irreducible supercuspidal representation of $Sp(W_2)$ mapping to $\pi_0$ under the theta correspondence. Then the preservation principle tells us that there exists an irreducible supercuspidal representation $\pi_2$ of $O(V_{6}^{-})$ corresponding to $\pi_1$ and that there exists some irreducible supercuspidal representation $\pi_3$ of $Sp(W_{10})$ corresponding to $\pi_2 \otimes \text{sgn}$. We can continue in this fashion, producing a chain of supercuspidal representations for our sequence.

Note that if we start at $Sp(W_2)$ with the representations $c \text{-Ind}_{Sp(L)}^{Sp(W_2)} R'(\theta)$ (here, $L$ is a lattice spanned by a standard symplectic basis of $W_2$), then we gets one chain of supercuspidals for each regular character $\theta$ of $\mu_{2+1} \subset SL(2, \mathbb{F})$. We will look at this chain in more depth in Section 3.4 and Section 4.2.
3.3 The preservation principle over a finite field

Although the theta correspondence is not 1-1 in general for dual reductive pairs over \( \mathbb{F} \), it still makes sense to discuss the smallest dimension at which a particular representation occurs in a Witt tower. The following result from [9] is the analogue of Theorem 2 for the finite theta correspondence. Again, we restrict to the case of symplectic/even orthogonal pairs for simplicity.

**Theorem 3** (Pan). Let \( \zeta \) be an irreducible cuspidal representation of \( G(\mathbb{v}) \), and let \( \{ \mathbf{v}_{\text{an}}^+ \oplus \mathbf{v}_{k,k}^+ | k \geq 0 \} \) and \( \{ \mathbf{v}_{\text{an}}^- \oplus \mathbf{v}_{k,k}^- | k \geq 0 \} \) be two Witt towers. Let \( m_{\pm}(\zeta) \) denote the smallest dimension of \( \mathbf{v}_{\text{an}}^\pm \oplus \mathbf{v}_{k,k}^\pm \) such that \( \zeta \) occurs in the theta correspondence for \( (G(\mathbb{v}), G(\mathbb{v}_+^\pm \oplus \mathbf{v}_{k,k}^\pm)) \).

(a) If \( G(\mathbb{v}) \) is symplectic and \( \mathbf{v}_{\text{an}}^+ \) and \( \mathbf{v}_{\text{an}}^- \) quadratic spaces with \( \dim \mathbf{v}_{\text{an}}^+ = 0 \) and \( \dim \mathbf{v}_{\text{an}}^- = 2 \), then \( m_{\pm}(\zeta) + m_{\mp}(\zeta) = 2 \dim \mathbf{v} + 2 \).

(b) If \( G(\mathbb{v}) \) is even orthogonal and \( \mathbf{v}_{\text{an}}^+ \) and \( \mathbf{v}_{\text{an}}^- \) are symplectic with dimension 0, then \( m_{\pm}(\zeta) + m_{\mp}(\zeta) = 2 \dim \mathbf{v} \).

Unlike the \( p \)-adic situation, we do not have a finite field analogue of Proposition 1; it does not help that the theta correspondence is not 1-1.

In Section 4, we will consider dual reductive pairs of the form \( (G(\mathbb{v}) \times G(\mathbb{w}), G(\mathbb{v}') \times G(\mathbb{w}')) \), where \( (G(\mathbb{v}), G(\mathbb{v}')) \) and \( (G(\mathbb{w}), G(\mathbb{w}')) \) are both symplectic/orthogonal pairs or both orthogonal/symplectic pairs. In this case, we say that a representation \( (\zeta \otimes \omega) \otimes (\zeta' \otimes \omega') \) of \( (G(\mathbb{v}) \times G(\mathbb{w})) \times (G(\mathbb{v}') \times G(\mathbb{w}')) \) is a first occurrence if \( \zeta \otimes \zeta' \) and \( \omega \otimes \omega' \) are both first occurrences.

3.4 Main example: The preservation principle for \( (\text{Sp}(2), O^+(4)) \)

We examine how the preservation principle interacts with the Deligne-Lusztig representations \( R'(\theta) \otimes \text{Ind}_{O^-(2)}^{O^+(2)}(\theta) \) of \( \text{Sp}(2) \). Since these representations are cuspidal. As in Theorem 3, we consider the Witt towers with \( \mathbf{v}_{\text{an}}^+ = 0 \) and \( \mathbf{v}_{\text{an}}^- \) the 2-dimensional anisotropic quadratic space (this is the quadratic extension \( \mathbb{v} / \mathbb{w} \) with the norm form). By Theorem 1, for any regular character \( \theta \) of \( \mu_{q+1} \), \( R'(\theta) \otimes \text{Ind}_{O^-(2)}^{O^+(2)}(\theta) \) occurs in the theta correspondence for the pair \( (\text{SL}(2), O^-(2)) \) (this can be seen from Theorem 1). Since \( O^-(2) \cong G(\mathbf{v}_{\text{an}}^-) \), this must be a first occurrence, i.e. \( m_{\mp}(R'(\theta)) = 2 \). Hence, Theorem 3 implies that \( m_{\mp}(R'(\theta)) = 4 \), so the first occurrence of \( R'(\theta) \) in the Witt tower \( \{ \mathbf{v}_{\text{an}}^+ \oplus \mathbf{v}_{k,k}^+ | k \geq 0 \} \) occurs in the theta correspondence for \( (\text{SL}(2), O^+(4)) \). By the discussion at the end of Section 2.2, there is some irreducible constituent \( \rho \) of \( R_{\text{Sp}(2) \times \text{SO}(4)}^{\text{SO}(2) \times \text{SO}(2)}(\theta \otimes 1) \) that realizes this first occurrence. More precisely, if \( \rho = \sigma(\rho) \) (recall that \( \sigma \) is conjugation by an element of \( O^+(4) \) outside \( \text{SO}(4) \)), then either \( R'(\theta) \otimes \rho' \) or \( R'(\theta) \otimes \rho'' \) is a first occurrence. If \( \rho \neq \sigma(\rho) \), then \( R'(\theta) \otimes \text{Ind}_{\text{SO}(4)}^{O^+(4)}(\rho) \) is a first occurrence.
4 Compatibility

So far, we have discussed various concepts and results that occur over both $p$-adic fields and finite fields, so it is natural to ask whether there exists some compatibility between these settings.

4.1 A result of Pan

For our purposes, call a lattice $L \subset V$ good if $\pi L^* \subset L \subset L^*$, where $L^* = \{ v \in V | \langle v, x \rangle_V \in \mathcal{O}_F \}$. Here, we are actually making a choice; we will fix an additive character $\psi$ of $F$ with conductor $\mathcal{O}_F$. If $L$ is a lattice in $V$, we can pull back a representation $\zeta$ of $G(L/pL)$ to $G_L$ (the subgroup of $G$ that preserves $L$) and then take the compact induction $c\text{-Ind}_{G_L}^{G(V)}(\zeta)$ to obtain a representation of $G(V)$.

We can attempt to go the other way if $L$ is a good lattice in $V$. Define

$$G_{L,0^+} := \{ g \in G(V) | (g-1)L^* \subset L, (g-1)L \subset \varpi L^* \}.$$ 

Then $G_{L,0^+}$ is a normal subgroup of $G_L$, and one can show that $G_L/G_{L,0^+}$ is a finite reductive group. A depth zero minimal K-type for $G(V)$ is a pair $(G_L, \zeta)$, where $\zeta$ is an irreducible representation of $G_L/G_{L,0^+}$. We say that an irreducible admissible representation $(\pi, W)$ of $G(V)$ contains a minimal K-type if the fixed vectors $W^{G_L,\pi^*}$ form a nonzero subspace and the resulting representation of $G_L/G_{L,0^+}$ on $W^{G_L,\pi^*}$ contains $\zeta$. If $\pi$ contains a depth zero minimal K-type, we say that $\pi$ is depth zero. We can obtain depth zero representations via compact induction. For any irreducible cuspidal representation $\zeta$ of $G_L/G_{L,0^+}$, it is known that $c\text{-Ind}_{G_L}^{G(V)}(\zeta)$ is an irreducible supercuspidal representation of $G(V)$ containing the minimal K-type $(G_L, \zeta)$.

We state a key result of [9] that establishes compatibility between the local theta correspondence and the finite theta correspondence for depth zero representations and minimal K-types they contain. For simplicity, we specialize to the case of symplectic/even orthogonal pairs.

It can be shown that if $(G(V), G(V'))$ is a dual reductive pair over $F$, then $(G_L/G_{L,0^+}, G_L'/G_{L',0^+}) = (G(L^*/L) \times G(L/\varpi L^*), G(L^*/L') \times G(L'/\varpi L'^*))$ is a dual reductive pair over $\mathfrak{g}$ in a natural way. Again, we state the following result only for symplectic/even orthogonal pairs, although it is stated more generally in [9] for any split dual reductive pair (a dual reductive pair $(G(V), G(V'))$ is split if the metaplectic covers $G(V) \to G(V)$ and $G(V') \to G(V')$ admit sections). Recall that the local theta correspondence depends on a choice of sections; part (b) is only true for a specific choice.

**Theorem 4** (Pan). Consider a symplectic/even orthogonal or even orthogonal/symplectic pair $(G(V), G(V'))$.

(a) Let $\pi$ (resp. $\pi'$) be an irreducible supercuspidal representation of $G(V)$ (resp. $G(V')$) such that $\pi \otimes \pi'$ is a first occurrence in the theta correspondence for $(G(V), G(V'))$. Suppose $\pi$ has a minimal K-type $(G_L, \zeta)$.
for some good lattice $L \subset V$. Then $\pi'$ has a minimal K-type $(G_{L'}, \zeta')$ such that $\zeta \otimes \zeta'$ is a first occurrence in the theta correspondence for the finite dual reductive pair $(G_L/G_{L,0^+}, G_{L'}/G_{L',0^+})$ with $\zeta, \zeta'$ cuspidal.

(b) Suppose $\zeta \otimes \zeta'$ is a first occurrence for the finite dual reductive pair $(\overline{G_L/G_{L,0^+}, G_{L'}/G_{L',0^+}})$ with $\zeta, \zeta'$ cuspidal. There exist sections $G(V) \to G(V)$ and $G(V') \to G(V')$ depending on $L$ and $L'$ such that $c \cdot \text{Ind}_{G_L}^{G(V)}(\zeta) \otimes c \cdot \text{Ind}_{G_{L'}}^{G(V')}((\zeta'))$ is a first occurrence for $(G(V), G(V'))$ with respect to this splitting.

4.2 Main example: Compatibility between $p$-adic $(\text{Sp}(V_2), O(V_6^+))$ and finite $(\text{SL}(2), O^-(2) \times O^+(4))$

We combine Theorem IV with the discussion in Section 2 and Section 3 to obtain a description of part of the theta correspondence for the pair $(\text{Sp}(W_2), O(V_6^-))$ considered in Section 3.2. Here, $W_2 = F^2$, and $V_6^- = E \oplus V_{2,2}$ ($E/F$ is the unramified quadratic extension with quadratic form $\varpi \text{Nm}_{E/F}$, and $V_{2,2}$ is a direct sum of two hyperbolic planes over $F$). We consider the good lattices $L := \mathcal{O}_F^2 \subset W_2$ and $L' := \mathcal{O}_E \oplus L_{2,2} \subset V_6^-$, where $L_{2,2}$ is a direct sum of two hyperbolic planes over $O_F$. Then $L^* = \mathcal{O}_F^2 \subset W_2$, so $G_L/G_{L,0} \cong \text{SL}(2, f)$, and $L'^* = \varpi^{-1} \mathcal{O}_E \oplus L_{2,2}$, so $G_{L'}/G_{L',0} \cong O^-(2) \times O^+(4)$. Hence, we have a finite dual reductive pair $(G_L/G_{L,0^+}, G_{L'}/G_{L',0^+}) \cong (\text{SL}(2), O^-(2) \times O^+(4))$, where the duality pairs the trivial group with $O^-(2)$ and $\text{SL}(2, f)$ with $O^+(4)$.

For the pair $(1, O^-(2))$, the Weil representation is trivial by definition, so $1$ is the only $O^-(2)$-representation occurring in the theta correspondence. Note that $1$ is a cuspidal representation of $O^-(2)$ because its connected component $SO^-(2)$ has no proper parabolic subgroups. For the pair $(\text{SL}(2), O^+(4))$, let $\rho$ be an irreducible constituent of the Deligne-Lusztig virtual character $R_{SO^-(2) \times SO^-(2)}^{SO^+(4)}(\theta \otimes 1)$ such that $R'(\theta) \otimes \rho$ occurs in the theta correspondence for $(\text{SL}(2), SO^+(4))$. Since $SO^-(2) \times SO^-(2)$ is an anisotropic torus, the virtual character $R_{SO^-(2) \times SO^-(2)}^{SO^+(4)}(\theta \otimes 1)$ must be cuspidal by [11] Proposition 6.3, in the sense that the (signed) sum of the Jacquet modules of its irreducible constituents must be 0 for any proper parabolic subgroup. Since the stabilizer of the Weyl group action on the character $\theta \otimes 1$ of $SO^-(2) \times SO^-(2)$ has size 2 (the nontrivial element inverts the second copy of $SO^-(2)$), there are exactly two irreducible constituents of $R_{SO^-(2) \times SO^-(2)}^{SO^+(4)}(\theta \otimes 1)$. Either both are cuspidal, or both are not. Unfortunately, we do not know the answer (please contact me if you do), so we consider both possibilities.

4.2.1 Components are not cuspidal

If neither irreducible constituent is cuspidal, then this is where the paper ends, since this means that the first occurrence of cuspidals for $(\text{SL}(2), SO^+(4))$ occurs outside the uniform part of the Weil representation, the only part we have
a concrete description of. We have reason to believe that this is indeed the case, due to the following lemma.

**Lemma 2.** (a) The group $\text{SO}^+(4)$ is isomorphic to the group $\text{SO}(V)$, where $V$ is the space of $2 \times 2$-matrices over $\mathbb{F}$ with quadratic form $x \mapsto \det x$.

(b) There is a surjection of algebraic groups $\text{SL}(2) \times \text{SL}(2) \to \text{SO}^+(4)$ with kernel \{±(1, 1)\}.

**Proof.** (a) It suffices to check that the quadratic form on $V$ has determinant 1. With respect to the natural basis of $V$, the quadratic form has matrix

\[
\begin{pmatrix}
1 & -1 & 0 & 0 \\
-1 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix},
\]

which has determinant 1.

(b) The group $\text{SL}(2) \times \text{SL}(2)$ acts on $V$ by $(g, h)x \mapsto gxh^{-1}$. This preserves the quadratic form, so $\text{SL}(2) \times \text{SL}(2)$ acts by elements of $\text{O}(V)$. Moreover, $\text{SL}(2) \times \text{SL}(2)$ is a connected algebraic group, so it lands in the connected component $\text{SO}(V)$.

We prove the fact about the kernel by considering the following cases. Suppose $(g, h)$ is in the kernel of $\text{SL}(2) \times \text{SL}(2) \to \text{SO}(V)$.

- $g \neq h$: The identity matrix 1 ∈ $V$ gets taken by $(g, h)$ to $gh^{-1} \neq 1$.
- $g = h$: If $g = h$ is not in the center of $\text{SL}(2)$, then $gxh^{-1} \neq x$ for any $x \in V$ that does not commute with $g = h$. The center of $\text{SL}(2)$ is ±1, so the kernel is \{±(1, 1)\}.

To finish, the map $\text{SL}(2) \times \text{SL}(2) \to \text{SO}(V)$ is a surjection of algebraic groups because both groups are 6-dimensional and connected and the map has finite kernel.

Lemma 2 shows that $\text{SL}(2) \times \text{SL}(2)$ and $\text{SO}^+(4)$ are not too far from one another. Moreover, $R_{\text{SO}^+(4)}(\theta \otimes 1) = R'((\theta) \otimes R_{\text{SL}(2)}^\text{SL}(2)) \otimes 1 = R'((\theta) \otimes \text{St} - R'((\theta) \otimes 1$.

Since the Steinberg representation $\text{St}$ and the trivial representation 1 are not cuspidal $\text{SL}(2)$-representations, $R'((\theta) \otimes \text{St}$ and $R'((\theta) \otimes 1$ are not cuspidal $\text{SL}(2)$-representations.

Since we have an isogeny $\text{SL}(2) \times \text{SL}(2) \to \text{SO}^+(4)$, we have good reason to believe that the components of $R_{\text{SO}^+(4)}(\theta \otimes 1)$ are not cuspidal. However, there are a few details that need to be ironed out. For instance, we are not sure how the embeddings $\text{SO}^-(2) \times \text{SO}^-(2) \to \text{SO}^+(4)$ and $\text{SO}^-(2) \times \text{SO}^-(2) \to \text{SL}(2) \times \text{SL}(2)$ are compatible. Thus, we do not know for sure.
4.2.2 Components are cuspidal

Now for the rest of this paper, we assume that the irreducible constituents of \( R_{SO^+(4)} \times SO^-(2) \times SO^-(2) (\theta \otimes 1) \) are cuspidal. We are well aware that this might be a total waste of time, due to the discussion in the above subsubsection. However, we do not know for sure.

Under our assumption of cuspidality, we see that the finite theta correspondence pairs \( R'(\theta) \) with some irreducible cuspidal \( O^+(4) \)-representation \( \xi \) according to which of the following cases occurs (see Section 2.3):

- \( \sigma(\rho) = \rho: \xi := 1 \otimes \rho \) or \( \xi := 1 \otimes \rho' \) (at least one occurs).
- \( \sigma(\rho) \neq \rho: \xi := 1 \otimes \text{Ind}_{SO^+(4)}^{SO^+}(\rho) \).

By the finite preservation principle (Theorem 3), \( R'(\theta) \otimes \xi \) is a first occurrence of cuspidal representations for the pair \((SL(2), O^-(2) \times O^+(4))\). Now we return to the \( p \)-adic dual reductive pair \((\text{Sp}(W_2), O(V_6))\). On the \( \text{Sp}(W_2) \cong SL(2, F) \) side, we consider the depth zero irreducible supercuspidal representations of the form \( c \cdot \text{Ind}_{SL(2, O_F)}^{SL(2, F)}(R'(\theta)) \), where \( \theta \) is a regular character of \( \mu_{q+1} \subset SL(2, F) \). Since \( R'(\theta) \otimes \xi \) is a first occurrence of cuspidal representations for the finite pair \((SL(2), O^-(2) \times O^+(4))\), Theorem 4(b) produces a first occurrence of supercuspidal representations \( c \cdot \text{Ind}_{SL(2, O_F)}^{SL(2, F)}(R'(\theta)) \otimes \pi' \) for the \( p \)-adic pair \((SL(2, F), O(V_6))\) (for choices of sections depending on our good lattices), where \( \pi' := c \cdot \text{Ind}_{O(L')}^{O(V_6)}(\xi) \). Because the local theta correspondence is 1-1, \( \pi' \) is uniquely determined by \( c \cdot \text{Ind}_{SL(2, O_F)}^{SL(2, F)}(R'(\theta)) \). In particular, it does not depend on choice of cuspidal \( O^+(4) \)-representation \( \xi \); we know that there are multiple choices of \( \xi \) because \( \theta \otimes 1 \) is not a regular character of \( SO^-(2) \times SO^-(2) \subset O^+(4) \).

In other words, although we are making choices on the finite level, they all result in the same supercuspidal representation on the \( p \)-adic level after applying compact induction.

5 Continuation of the main example

This entire section explains what we would like to happen. If all goes well, the supercuspidal chain from Section 3.2 admits a concrete description in terms of the uniform part of the finite Weil representation (using Theorem 1). This could very well be false.

In the previous sections, we applied results of Srinivasan and Pan to match first occurrences of cuspidal representations for the finite pair \((\text{Sp}(2), O^-(2) \times O^+(4))\) with first occurrences of supercuspidal representations for the \( p \)-adic pair \((\text{Sp}(W_2), O(V_6^-))\) from Section 3.2. This assumed the cuspidality of the irreducible constituents of \( R_{SO^-(2) \times SO^-(2)}^{SO^+(4)}(\theta \otimes 1) \).

Using the roughly the same methods, we hope to be able to obtain descriptions for the entire chain of supercuspidals from Section 3.2 in terms of cuspidal
representations of a corresponding chain of finite reductive groups. In this section, we describe partial progress towards this goal (unfortunately, we have not been able to flesh the details out completely, starting with the cuspidality of the irreducible constituents of our Deligne-Lusztig virtual characters).

To refresh the reader’s memory, the first couple terms of the chain of $p$-adic groups are

$$O(V^+_2), Sp(W_2), O(V^-_6), Sp(W_{10}), O(V^+_{18}), Sp(W_{26}), O(V^-_{38}), \ldots,$$

where the dimensions are determined by Theorem 2. Here, $V^+_r$ (resp. $V^-_r$) denotes the quadratic space $E \oplus V_{(r-2)/2,(r-2)/2}$ with the quadratic form $Nm_{E/F}$ (resp. $\varpi Nm_{E/F}$) on $E$ if $r \equiv 2 \pmod{8}$ (resp. $r \equiv 6 \pmod{8}$), and $Sp(W_r)$ denotes a symplectic space of dimension $r$. We would like to match this chain of $p$-adic reductive groups with the chain of finite reductive groups

$$1 \times O^- (2), 1 \times Sp(2), O^- (2) \times O^+ (4), Sp(4) \times Sp(6),$$

$$O^+ (8) \times O^- (10), Sp(12) \times Sp(14), O^- (18) \times O^+ (20), \ldots,$$

where the dimensions are determined by Theorem 3. We would like to find a chain of cuspidal representations of these finite reductive groups, such that each consecutive pair is a first occurrence.

The following steps correspond to Section 2.3, Section 3.4, and Section 4.2, in that order.

(1) We would like to apply the Srinivasan-Pan formula (Theorem 1) to match representations in our chain of finite reductive groups. Unfortunately, the representation theory of the groups in our chain is less well-understood than that of $SL(2)$. The difficulty lies in the fact that the Deligne-Lusztig virtual characters could have overlaps, so it is difficult to pick out irreducible constituents and, moreover, irreducible cuspidal constituents. For $SL(2)$, this does not occur because the Deligne-Lusztig representations $R' (\theta)$ are bona fide irreducible cuspidal representations. However, as we go up the chain, we no longer have regular characters, so we lose irreducibility. Ideally, we should be able to obtain a sequence of cuspidals occurring in the following sequence of Deligne-Lusztig virtual characters (here, we use $SO$ instead of $O$ for simplicity):

$$\theta, R' (\theta), 1 \otimes R_{SO^- (2) \times SO^- (2)}^{SO^+ (4)} (\theta \otimes 1), R_{\mu_{q+1} \times Sp(2)}^{Sp(4)} (1 \otimes 1) \otimes R_{\mu_{q+1} \times Sp(4)}^{Sp(6)} (\theta \otimes 1)$$

$$R_{SO^- (2) \times SO^- (6)}^ {SO^- (8)} (1 \otimes 1) \otimes R_{\mu_{q+1} \times Sp(12)}^{SO^- (10)} (\theta \otimes 1),$$

$$R_{\mu_{q+1} \times Sp(10)}^{Sp(12)} (1 \otimes 1) \otimes R_{\mu_{q+1} \times Sp(12)}^{Sp(14)} (\theta \otimes 1),$$

$$R_{SO^- (2) \times SO^- (16)}^{SO^- (18)} (1 \otimes 1) \otimes R_{SO^- (2) \times SO^- (18)}^{SO^- (20)} (\theta \otimes 1), \ldots.$$
(2) Assuming we can find cuspidal theta lifts in the above chain of Deligne-Lusztig virtual characters, the application of the finite preservation principle is identical to the application in Section 3.4.

(3) We can pick good lattices $L \subset V_r^\pm$ (resp. $L' \subset W_r$) such that the resulting finite reductive groups $G(L^*/L) \times G(L/\varpi L^*)$ (resp. $G(L'^*/L') \times G(L'/\varpi L'^*)$) are the groups $O^\pm(r/2 - 1) \times O^\mp(r/2 + 1)$ (resp. $\text{Sp}(r/2 - 1) \times \text{Sp}(r/2 + 1)$) in our sequence. Once we have our good lattices, we can apply Theorem 4 to match the finite theta correspondence with the local theta correspondence. We pick good lattices as follows:

- $V_r^+$: Recall that $V_r^+ = E + V_{(r-1)/2,(r-1)/2}$, where $E/F$ is the unique unramified quadratic extension and $V_{1,1}$ is the hyperbolic plane. Pick $a_1, \ldots, a_{r/2-1}, b_1, \ldots, b_{r/2-1}$ such that $a_i, b_i$ is a standard basis of the $i$th hyperbolic plane. Let

$$L := \mathcal{O}_E + \sum_{i=1}^{(r-2)/4} (\mathcal{O}_F a_i + \mathcal{O}_F b_i) + \sum_{i=(r+2)/4}^{r/2-1} (\mathcal{O}_F a_i + \varpi \mathcal{O}_F b_i).$$

Then

$$L^* = \mathcal{O}_E + \sum_{i=1}^{(r-2)/4} (\varpi^{-1} \mathcal{O}_F a_i + \mathcal{O}_F b_i) + \sum_{i=(r+2)/4}^{r/2-1} (\mathcal{O}_F a_i + \mathcal{O}_F b_i),$$

so $G(L^*/L) \cong O^+(r/2-1)$, and $G(L/\varpi L^*) \cong O^-(r/2+1)$, as desired.

- $V_r^-$: Use the same notation as the previous case. Let

$$L := \mathcal{O}_E + \sum_{i=1}^{(r-6)/4} (\mathcal{O}_F a_i + \varpi \mathcal{O}_F b_i) + \sum_{i=(r-2)/4}^{r/2-1} (\mathcal{O}_F a_i + \varpi \mathcal{O}_F b_i).$$

Then

$$L^* = \varpi^{-1} \mathcal{O}_E + \sum_{i=1}^{(r-6)/4} (\varpi^{-1} \mathcal{O}_F a_i + \mathcal{O}_F b_i) + \sum_{i=(r-2)/4}^{r/2-1} (\mathcal{O}_F a_i + \varpi \mathcal{O}_F b_i),$$

so $G(L^*/L) \cong O^-(r/2-1)$, and $G(L/\varpi L^*) \cong O^+(r/2+1)$, as desired.

- $W_r$: Let $a_1, \ldots, a_{r/2}, b_1, \ldots, b_{r/2}$ be a standard symplectic basis of $W_r$. Let

$$L' := \sum_{i=1}^{(r-2)/4} (\mathcal{O}_F a_i + \varpi \mathcal{O}_F b_i) + \sum_{i=(r+2)/4}^{r} (\mathcal{O}_F a_i + \mathcal{O}_F b_i).$$

Then

$$L'^* = \sum_{i=1}^{(r-2)/4} (\varpi^{-1} \mathcal{O}_F a_i + \mathcal{O}_F b_i) + \sum_{i=(r+2)/4}^{r/2} (\mathcal{O}_F a_i + \mathcal{O}_F b_i),$$

so $G(L'^*/L') \cong \text{Sp}(r/2 - 1)$, and $G(L'/\varpi L'^*) \cong \text{Sp}(r/2 + 1)$, as desired.
References


